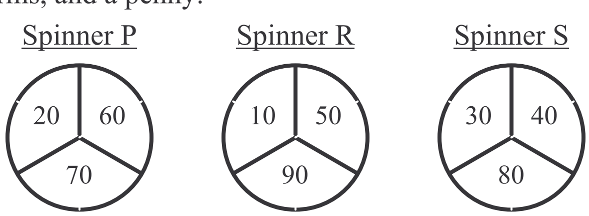
Probability Experiments for Student Investigation

Hands-on experiments often serve both to spark students' interest and to aid their understanding. Here is a probability project that has repeatedly been a hit with our students. The experiment is an easy-to-create modification of the non-transitive dice game designed by noted statistician Bradley Efron [1]. We recommend that students explore the game from (at least) three points of view: a physical experiment, a computer simulation, and a symbolic analysis.

Project Overview: Students investigate a simple game using three spinners (pictured below). They first experiment with paper spinners and then use a computer to simulate numerous repetitions of the game. After they have estimated the probability through experimentation, students analyze the mathematics behind the game. One hook for the students is that the results are surprising. Another is that computer simulations allow students to run enough trials so that their mathematical calculations are reflected in their experimental results. When this happens, many students feel mathematically powerful. By proceeding in stages from the concrete physical spinners through the computer model to the mathematical abstraction, we help ensure that the abstraction is meaningful to the students. This gradual approach to abstraction in one problem often enables students to move easily toward abstraction in future applications.

The Audience: We have used this project (in appropriately modified forms) with grade 3-8 students attending mathematics enrichment workshops, as an activity for middle school mathematics and science clubs, as a topic in college courses for pre-service teachers, and for workshops for in-service teachers.

Project Details: The students are divided into teams of three and each group is given the set of three spinners shown below, two paper clips, data recording forms, and a penny.



Part 1—Paper Spinners: The spinners and data recording form can be easily made or downloaded from www.as.ysu.edu/~math/faculty/pollack.htm. The paper spinners work by unfolding one bend of the paper clip to act as a pointer. The tip of the pencil is then inserted through the loop of the paper clip and placed in the center of the spinner. Students quickly become proficient at flicking the paper clip so that the pointer whirls around the pencil until landing in one of the sectors (if the pointer lands directly on a dividing line it should be re-spun). These spinners are low-cost, easy to reproduce, don't take up storage space, and perform extremely well.

For the first experiment, we ask one team member to take spinner P, another to take spinner R and the third to record the results. The two students simultaneously flick their spinners and the spinner landing on the higher number wins. Each group conducts 50 trials of the experiment and writes their results on the blackboard. For most of the groups, spinner P will beat spinner R in the 50-spin round (there is a 74% chance) and almost surely the pooled results from the class will show spinner P defeating spinner R (99.3% chance with 500 class spins). We decide, as a class, that P is better than R and briefly discuss that, despite its superiority, P will lose some of the 50-spin games.

In the second experiment, the student with spinner P becomes the recorder, the student who had spinner R keeps it, and the student who was the recorder the plays with spinner S. Again each group does 50 trials and class results are tabulated. With the same probabilities as the first experiment, it turns out that spinner R defeats spinner S.

The final experiment is a match between P and S. The student that has not yet been recorded serves in that role. The other two players flip a coin and the winner has the choice between P and S. Since P beats R and R beats S, many students will conclude that P is the best of the three. Contrary to inclination, the results of the experiment will show (with the same probabilities as above) that S beats out P. Spinner power in this game is not transitive!

Purposeful activity: The spinning game is fun (for most audiences) and the surprising result usually generates interest. However, just as important as these motivational factors is its role in enhancing conceptualization of the key ideas. For example, working with the physical spinners makes the sample space of the experiment concrete. Students unfamiliar with these ideas might conclude that three possible outcomes for spinner P and three outcomes for spinner R combine to give six possibilities altogether. However, the students will see all nine possibilities by doing the experiment themselves and recording the results. This concrete introduction gives meaning to our later representation of the problem using a table of equally likely outcomes or a tree diagram.

Part 2—Computer Simulations: After using the paper spinners to collect data themselves, students use Microsoft Excel to generate 2000 trials of the experiment. Exactly how the computer is used depends on the audience. We ask college students to build the spreadsheet themselves, but we provide a pre-built spreadsheet for young children. You can download the Excel file from the previously mentioned website. If you are interested in the computer simulation, but not fluent in Excel, having the spreadsheet in front of you will make the following description easier to read. For a short activity for a middle school math club, you could skip the computer simulation and move directly to part 3.

To simulate random events, we use the pseudo-random number generator in Excel. This function is called using the command =rand(), which results in a decimal in the interval [0,1). If, for example, we multiply this function by six and add one, =6*rand()+1, we will get a decimal in the interval [1,7). Then, by applying the greatest integer function, =int(6*rand()+1), we could simulate the rolling of a fair die. (This can be accomplished in one step with the function =randbetween(1,6), but this command, part of the "Analysis ToolPak", is frequently not installed.) Ask students to try out these commands before proceeding. Show them that the function key F9 causes the spreadsheet to recalculate and generate new random numbers.

Once we have the random number, we use the =choose() command to simulate the spinners. The syntax of this command is =choose(exp,o1,o2,...,oN), where exp is an expression or cell reference that returns a real number in [1,N+1), and o1,o2,...,oN are the outputs we desire to produce for each of the N possibilities for the integer part of exp. We can simulate spinner P with the command =choose(int(3*rand()+1),20,60,70). Type this command once and copy it down 50 rows to create a column of spinner P results that looks like the sheet on which we recorded our data from the paper spinners. Indeed, we designed the layout of the data-recording sheet to make the computer simulation seem like a natural extension of the earlier hands-on experience. This is helpful for younger students. The commands to simulate spinners R and S involve obvious modifications. Ask students to create 2000 spinner P results in column C and 2000 spinner R results in column E.

We use the =if() command to determine the winner of each spin. The syntax is =if(condition,oT,oF), where oT is the output generated if the condition is true and oF is the output generated if the condition is false. Thus the command =if(C3>E3,"P","R") will produce the output P if the number in cell C3 is greater than the number in cell E3 and R if it is not. Finally, we use the =countif() command to count the number of P's and R's in the winner list. The command, as we used it, is =countif(range,target), where range is the set of cells in which we want to count and target describes the item to be counted. Thus the command =countif(C3:C2002,"P") returns the number of cells between C3 and C2002 that evaluate to P. Finally, we compute the fraction of spins in which spinner P wins. By pressing F9 several times, students get a feel for typical winning percentages.

In a second and third worksheet within our Excel file, we calculate the relative frequency of R beating S and S beating P. With experimental results in hand, we next ask the students to calculate the actual probability.

Part 3—Mathematical Analysis: Students calculate the probability of P beating R by making a table of equally likely outcomes.

		Spinner R		
		10	50	90
Spinner P	20	P	R	R
	60	P	P	R
	70	P	P	R

From the table above, we see that Spinner P is the winner 5 out of 9 times. Similarly, students find that the probability of Spinner R beating Spinner S is 5/9. However, the probability of Spinner P beating Spinner S is 4/9. In a group discussion at the end of the project, we ask the students to explain their findings. We also ask if they have any idea why the spinners are named P, R, and S. We usually get a smile when they recall the children's game "Paper, Rock, Scissors". You might remember that paper covers rock, rock dulls scissors, and scissors cuts paper.

1. Martin Gardner, *Mathematical Games*, Scientific American, **223** (6) (1970), 110-114.

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